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q-ADIC SPECTRAL ANALYSIS OF SOME ARITHMETIC SEQUENCES

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Abstract. The metric group action of $\mathbb{N}$ endowed with the q-adic addition to general base q is investigated. This gives rise to q-adic spectral analysis, which is applied to q-multiplicative sequences. Multiplicative sequences in a base q' prime to q are also studied.

Contents

1. Introduction

1.1. Generalities

Let $q = (q_r)_{r\geq 0}$ be a given sequence of natural numbers such that $q_0 = 1$ and $q_r \geq 2$ for $r \geq 1$. Each natural number can be written in a single way: $n = \sum_{r\geq 0} e_r(n)p_r$ where $e_r(n) \in \{0, \ldots, q_{r+1} - 1\}$, and $p_r = q_0 \ldots q_r$ for each $r \in \mathbb{N}$. The integer $e_r(n)$ is called the rth digit of n in base q.

Referring to Coquet [3], we will say that a sequence u of complex numbers is q-multiplicative if $u(0) = 1$ and $u(ap_r + b) = u(ap_r)u(b)$, for any natural numbers a, b, r, such that $b < p_r$.

The spectral analysis of these sequences was the subject of numerous researches, in particular [2, 7, 12]; the classical dynamical point of view is detailed in [9, 10]. Our aim is to carry on the harmonic study of these sequences in the case of a general base as begun by Coquet in [4] for the 2-adic addition.

1.2. Group G_q, G_q-flows

Let q be given as before. The q-adic addition denoted $\oplus_q$ (or more simply $\oplus$) is defined on $\mathbb{N}$ by

$$e_r(n \oplus_q m) = e_r(n) + e_r(m) \bmod q_{r+1} \quad \text{for each } r \geq 0.$$

$(\mathbb{N}, \oplus_q)$ is a group denoted by G_q, isomorphic to the weak product $\coprod_{r=1}^{\infty} \mathbb{Z}/q_r\mathbb{Z}$, whose dual is isomorphic to the compact product $\prod_{r=1}^{\infty} (\mathbb{Z}/q_r\mathbb{Z})$ (cf. [6]).

Let us denote $\mathrm{e}(\cdot) = \exp(2\mathrm{i}\pi(\cdot))$. The isomorphism between $\hat{G}_q$ and this product is given by the following expression of the characters of $\hat{G}_q$, defined for $n \in \mathbb{N}$ by

$$w_n(t) = \prod_{r \geq 0} \mathrm{e}(t_r e_r(n)/q_{r+1}),$$

where t is identified to the sequence $(t_r)_{r \geq 0}$. The function w_n is called a Walsh character in base q.

X will denote a compact metrizable space, β the σ-algebra of the Borel subsets of X, and $P(X)$ the set of the Borel probability measures on X.

Definition. Let G be a group. A G-flow on X, denoted $(G; X)$, is defined by a group morphism $T: G \to \operatorname{Aut} X$, from G to the group of the bicontinuous one-to-one maps on X.

We will only be interested in the G_q-flows, and we will choose the following definition.

Definition. A G_q-dynamical system or G_q-process $\chi = (G_q; X, \mu)$ is a G_q-flow $(G_q; X)$ with an element μ of $P(X)$ such that $\mu(T_g^{-1}B) = \mu(B)$ for each $g \in G_q$ and $B \in \beta$.

For instance, if we define the q-adic shift by $T_g((x_n)_{n \in \mathbb{N}}) = (x_{n \oplus g})_{n \in \mathbb{N}}$ with μ an element of $P(X)$, then $(G_q; X^{\mathbb{N}}, \mu_\infty)$ is a G_q-dynamical system.

1.3. Walsh correlations and Walsh spectral measures

Definition. Let $(G_q; X, \mu)$ be a G_q-dynamical system, and f an element of $L^2(X; \mu)$. The Walsh correlation of f, $\gamma_f: G_q \to \mathbb{C}$, is defined by

$$\gamma_f(m \ominus n) = (T_m f \mid T_n f) = \int_X T_m f \overline{T_n f} \, \mathrm{d}\mu$$

(the definition is correct as $(G_q; X, \mu)$ is a G_q-dynamical system).

The sequence $((\gamma_f(n))_{n \in \mathbb{N}}$ is G_q-positively definite and one checks $\gamma_f(0) = \|f\|_2^2$; according to the Bochner–Herglotz theorem, we have the following proposition.

Proposition. *If $f \neq 0$, there is a unique Borel measure (or even probability measure if $\|f\|_2 = 1$) on $\hat{G}_q$, denoted by λ_f, and such that*

$$\gamma_f(n) = \int_{\hat{G}_q} w_n(t)\lambda_f(\mathrm{d}t) \quad \textit{for each } n \in \mathbb{N}.$$

Proposition. *λ_f is the limit in the sense of weak convergence of the sequence of measures*

$$\lambda_f^N = p_N^{-1}\left[\int_X \left| \sum_{n<p_N} \overline{w_n(t)} f(T_n x)\right|^2 \mu(\mathrm{d}x)\right] h(\mathrm{d}t),$$

where h is the Haar probability measure on $\hat{G}_q$.

Proof. $(\lambda_f^N)_N$ is a bounded sequence of positive measures such that for each $k \in \mathbb{N}$, $k < p_N$,

$$\begin{aligned}
\lambda_f^N(w_k) &= \int_{\hat{G}_q} p_N^{-1} w_k(t)\left[\int_X \left| \sum_{n<p_N} \overline{w_n(t)} f(T_n x)\right|^2 \mu(\mathrm{d}x)\right] h(\mathrm{d}t) \\
&= p_N^{-1} \int_{\hat{G}_q} \sum_{n,p<p_N} \left[\int_X w_k(t)\overline{w_{p\ominus n}(t) f(x)} f(T_{p\ominus n}x)\mu(\mathrm{d}x)\right] h(\mathrm{d}t) \\
&= \sum_{p<p_N} \int_{\hat{G}_q} \left[\int_X w_k(t)\overline{w_p(t) f(x)} f(T_p x)\mu(\mathrm{d}x)\right] h(\mathrm{d}t) \\
&= \int_X \overline{f(x)} f(T_k x)\mu(\mathrm{d}x) = \gamma_f(k).
\end{aligned}$$

Hence the conclusion, as the w_n's constitute a base of $\mathscr{C}(\hat{G}_q)$. □

Remark. Let $\mathscr{C}_{N,a}$ be the cylinder $\{(x_n)_n \in \hat{G}_q;\ \forall n < N, x_n = a_n\}$, where $N \in \mathbb{N}$ and $a = (a_0, a_1, \dots)$ is a given element of $\hat{G}_q$. Using the same considerations as in the proof of the foregoing proposition, $h(\mathscr{C}_{N,a}) = p_N^{-1}$, and $\mathbf{1}_{\mathscr{C}_{N,a}}$ is orthogonal to w_p whenever $p \geq p_N$, we get

$$\lambda_f(\mathscr{C}_{N,a}) = p_N^{-1} \sum_{n<p_N} \overline{w_n(\mathscr{C}_{N,a})}\gamma_f(n),$$

where $w_n(\mathscr{C}_{N,a})$ is the constant value of w_n on $\mathscr{C}_{N,a}$. Since $\lambda_f(\{a\}) = \lambda_f(\bigcap_N \mathscr{C}_{N,a})$, we have

$$\lambda_f(\{a\}) = \lim_{N\to\infty} p_N^{-1} \sum_{n<p_N} \overline{w_n(a)}\gamma_f(n).$$

1.4. Asymptotical behaviour of the Walsh spectral measures

Theorem 1.1 below studies the asymptotical behaviour of the Walsh spectral measures when the base q varies. Let $(b_N)_N$ be a strictly increasing sequence of integers ≥ 2. In this section we consider bases of the form $(1, b_N, b_N, \dots)$, identified with the integer b_N. Let $(\chi_N = (G_{b_N}; X_N, \mu_N))_{N\in\mathbb{N}}$ be a family of G_{b_N}-processes on a sequence of Borel probability spaces $(X_N; \beta_N, \mu_N)$. To each sequence $(f_N)_{N\in\mathbb{N}}$, $f_N \neq 0$, of functions of $L^2(X_N; \mu_N)$ with correlations $\gamma_{f_N}^{(N)}$, a sequence of Borel measures $(\lambda_{f_N}^{(N)})_{N\in\mathbb{N}}$ is associated, where $\lambda_{f_N}^{(N)}$ is defined on $\hat{G}_{b_N}$. For each base q, we will denote by Ψ_q the continuous map from $\hat{G}_q$ to the torus $\mathbb{T}$ which associates to an element $(l_k)_{k\in\mathbb{N}}$ of $\hat{G}_q$ the class of $\sum l_k/p_{k+1} \bmod 1$ (with $l_k \in \{0, \ldots, q_{k+1}-1\}$). Let us denote by $\tilde{\lambda}_{f_N}^{(N)}$ the image of the measure $\lambda_{f_N}^{(N)}$ under Ψ_{b_N}.

1.1. Theorem. *With the previous notations, if $\chi = (X; T, \mu)$ is a dynamical system and f is an element of $L^2(X; \mu)$, $f \neq 0$, such that for each natural number k the sequence $(\gamma_{f_N}^{(N)}(k))_{N\in\mathbb{N}}$ converges to $\Gamma_f(k)$, where $\Gamma_f : \mathbb{Z} \to \mathbb{C}$ is the correlation of f in the classical way, then the sequence $(\tilde{\lambda}_{f_N}^{(N)})_{N\in\mathbb{N}}$ converges weakly to the spectral measure Λ_f of f with respect to χ.*

Proof. The sequence $(\tilde{\lambda}_{f_N}^{(N)})_{N\in\mathbb{N}}$ is a bounded sequence of positive measures on $\mathbb{T}$. So it is sufficient to check that, for each natural number k,

$$(*) \qquad \lim_{N\to\infty} \tilde{\lambda}_{f_N}^{(N)}(\mathrm{e}(k\cdot)) = \Lambda_f(\mathrm{e}(k\cdot)).$$

But

$$\begin{aligned}\tilde{\lambda}_{f_N}^{(N)}(\mathrm{e}(k\cdot)) &= \int_{\hat{G}_{b_N}} \mathrm{e}(k\Psi_{b_N}(t))\lambda_{f_N}^{(N)}(\mathrm{d}t)\\ &= \int_{\hat{G}_{b_N}} \mathrm{e}\Big(b_N^{-1}k \sum_{i\geq 0} b_N^{-i}t_i\Big)\lambda_{f_N}^{(N)}(\mathrm{d}t),\end{aligned}$$

where $t = (t_i)_{i\in\mathbb{N}}$ with $t_i < b_N$, $\forall i \in \mathbb{N}$, and $\Lambda_f(\mathrm{e}(k\cdot)) = \Gamma_f(k) = \lim_{N\to\infty} \gamma_{f_N}^{(N)}(k)$, where, if $k = \sum k_j b_N^j$

$$\begin{aligned}\gamma_{f_N}^{(N)}(k) &= \int_{\hat{G}_{b_N}} \prod_{j\geq 0} \mathrm{e}(b_N^{-1}k_jt_j)\lambda_{f_N}^{(N)}(\mathrm{d}t)\\ &= \int_{\hat{G}_{b_N}} \mathrm{e}(b_N^{-1}kt_0)\lambda_{f_N}^{(N)}(\mathrm{d}t) \qquad \text{if } k < b_N.\end{aligned}$$

And we have

$$\begin{aligned}&\left|\int_{\hat{G}_{b_N}} \mathrm{e}\Big(b_N^{-1}k \sum_{i\geq 0} b_N^{-i}t_i\Big)\lambda_{f_N}^{(N)}(\mathrm{d}t) - \int_{\hat{G}_{b_N}} \mathrm{e}(b_N^{-1}kt_0)\lambda_{f_N}^{(N)}(\mathrm{d}t)\right|\\ &\leqslant \int_{\hat{G}_{b_N}} \left|\mathrm{e}\Big(k \sum_{i>0} b_N^{-i-1}t_i\Big) - 1\right| \lambda_{f_N}^{(N)}(\mathrm{d}t)\\ &\leqslant \sup_{(t_n)_n\in\{0,\ldots,b_N-1\}^{\mathbb{N}}} \left|\mathrm{e}\Big(k \sum_{i>0} b_N^{-i-1}t_i\Big) - 1\right| \leqslant |\mathrm{e}(b_N^{-1}k) - 1|,\end{aligned}$$

which tends to 0 when N tends to $+\infty$, and therefore we have proved $(*)$. □

2. G_q-flows associated to a sequence

2.1. Definition

Let X be a compact metrizable space, $u=(u_n)_{n\in\mathbb{N}}$ an element of $Y=X^{\mathbb{N}}$ and Y_u the orbit closure of u under the action of the q-adic shift. Let $I(u)$ be the set of the measures of Y, limit points in the sense of weak convergence of the Dirac averages $(N^{-1}\sum_{n<N}\delta_{T_n u})_{N\in\mathbb{N}}$, and v an element of $I(u)$; the support of v is in Y_u.

Proposition. *$\chi_v=(G_q;\, Y_u,\, v)$ is a G_q-dynamical system.*

Proof. Let f be an element of $\mathscr{C}(Y_u)$, and J an infinite subset of $\mathbb{N}$ such that

$$v=*\text{-}\lim_{\substack{N\to\infty\\ N\in J}} N^{-1}\sum_{n<N}\delta_{T_n u}.$$

For each element g of G_q,

$$v(f)-v(T_g f)=\lim_{\substack{N\to\infty\\ N\in J}} N^{-1}\sum_{n<N} f(T_n u)-\lim_{\substack{N\to\infty\\ N\in J}} N^{-1}\sum_{n<N} f(T_{n\oplus g}u).$$

Assume that $p_l\leqslant g<p_{l+1}$. Then, for N large enough,

$$\operatorname{card}(\{n;\, n<N\}\cap\{n\oplus g;\, n<N\})\geqslant N-2p_{l+1}$$

and therefore

$$\left|N^{-1}\sum_{n<N} f(T_n u)-N^{-1}\sum_{n<N} f(T_{n\oplus g}u)\right|\leqslant N^{-1}\sup_{Y_u}|f|\cdot 2p_{l+1}.$$

Hence, by going to the limit, $v(f)=v(T_g f)$. □

2.2. The set $\mathscr{S}_q$

Consider the set $\mathscr{S}_q$ of the sequences $u:\mathbb{N}\to\mathbb{U}=\{z\in\mathbb{C};\,|z|=1\}$ such that

$$\lim_{N\to\infty} N^{-1}\sum_{n<N} u(n\oplus k)\overline{u(n)}$$

exists for each natural number k. Let $\boldsymbol{P}_0: Y_u\to\mathbb{U}$ be the projection on the first coordinate, and $v\in I(u)$. $\boldsymbol{P}_0$ is an element of $\mathscr{C}(Y_u)$, and its correlation in the sense of χ_v is given by

$$\forall k\in\mathbb{N},\quad \gamma_{\boldsymbol{P}_0}(k)=\int T_k\boldsymbol{P}_0\cdot\overline{\boldsymbol{P}_0}\,\mathrm{d}v=\lim_{N\to\infty} N^{-1}\sum_{n<N} u(n\oplus k)\overline{u(n)}.$$

Notice that $\gamma_{\boldsymbol{P}_0}$ does not depend on the choice of the elements of $I(u)$.

Definition. Let u be an element of $\mathscr{S}_q$. The Walsh correlation of u is the sequence of complex numbers denoted by γ^u and defined by

$$\gamma^u(k)=\lim_{N\to\infty} N^{-1}\sum_{n<N} u(k\oplus n)\overline{u(n)}.$$

Now, for $\chi=(G_q;\, Y_u, v)$, $v\in I(u)$, we use the results of Subsection 1.3 to conclude that there is a unique Borel probability measure on $\hat{G}_q$, denoted λ_{P_0}, such that for each natural number k,

$$\gamma_{P_0}(k)=\int_{\hat{G}_q} w_k(t)\lambda_{P_0}(\mathrm{d}t).$$

Definition. λ_{P_0} is called the *Walsh spectral measure of u* and is denoted by λ^u.

Proposition. (1) *If v is an element of $I(u)$, and J an infinite subset of $\mathbb{N}$ associated to v, then λ^u is the limit in the sense of weak convergence of the sequence of measures*

$$p_N^{-1}\left[\lim_{\substack{K\to\infty\\ K\in J}} K^{-1}\sum_{k<K}\left|\sum_{n<p_N}\overline{w_n(t)}u(n\oplus k)\right|^2\right]h(\mathrm{d}t),$$

and of the sequence of measures

$$p_N^{-1}\left|\sum_{n<p_N}\overline{w_n(t)}u_n\right|^2 h(\mathrm{d}t).$$

(2)
$$\lambda^u(\mathscr{C}_{N,a})=p_N^{-1}\sum_{n<p_N}\overline{w_n(\mathscr{C}_{N,a})}\gamma^u(n)$$
$$=p_N^{-2}\lim_{K\to\infty}K^{-1}\sum_{m<K}\left|\sum_{n<p_N}\overline{w_n(\mathscr{C}_{N,a})}u(n+mp_N)\right|^2.$$

(3) *For each element a of $\hat{G}_q$,*

$$\lambda^u(\{a\})=\lim_{n\to\infty}p_N^{-1}\sum_{n<p_N}\overline{w_n(a)}\gamma^u(n)$$
$$=\lim_{N\to\infty}p_N^{-2}\lim_{K\to\infty}K^{-1}\sum_{m<K}\left|\sum_{n<p_N}\overline{w_n(a)}u(n+mp_N)\right|^2.$$

Proof. (1) is easily checked on the Fourier coefficients.

(2):
$$\lambda^u(\mathscr{C}_{N,a})=p_N^{-1}\sum_{n<p_N}\overline{w_n(\mathscr{C}_{N,a})}\lim_{K\to\infty}K^{-1}p_N^{-1}\sum_{k<Kp_N}u(k\oplus n)\overline{u(k)}$$
$$=\lim_{K\to\infty}K^{-1}\sum_{m<K}p_N^{-2}\sum_{n<p_N}\sum_{p<p_N}\overline{w_n(\mathscr{C}_{N,a})}u((p+mp_N)\oplus n)\overline{u(p+mp_N)}$$
$$=\lim_{K\to\infty}K^{-1}\sum_{m<K}\left|p_N^{-1}\sum_{n<p_N}\overline{w_n(\mathscr{C}_{N,a})}u(n+mp_N)\right|^2,$$

since $w_n(\mathscr{C}_{N,a})=\overline{w_p(\mathscr{C}_{N,a})}w_{n\oplus p}(\mathscr{C}_{N,a})$ if $n<p_N$ and $p<p_N$,

(3) follows from (2). □

2.3. Properties of the Walsh correlations

For the Walsh correlations we are going to prove properties analogous to the ones of usual correlations.

2.1. Theorem. *For each element α of $\hat{G}_q$, we have*

$$\limsup_N N^{-1}\left|\sum_{n<N} u_n\overline{w_n(\alpha)}\right| \leqslant \lambda^u(\{\alpha\})^{1/2}.$$

Proof. Using the formula

$$(Ns)^{-1}\left|\sum_{n<Ns} a_n\right| \leqslant N^{-1}\sum_{m<N} s^{-1}\left|\sum_{n<s} a_{n+ms}\right|,$$

the Cauchy–Schwarz inequality, and (2) of the foregoing proposition, for a fixed natural number K we obtain

$$\begin{aligned}
&\left(\limsup_N N^{-1}\left|\sum_{n<N} u_n\overline{w_n(\alpha)}\right|\right)^2\\
&\leqslant\left(\limsup_N N^{-1}\sum_{m<N} p_K^{-1}\left|\sum_{n<p_K} u(n+mp_K)\overline{w_{n+mp_K}(\alpha)}\right|\right)^2\\
&=\left(\limsup_N N^{-1}\sum_{m<N}\left|p_K^{-1}\sum_{n<p_K} u(n+mp_K)\overline{w_n(\alpha)}\right|\right)^2\\
&\leqslant\lim_{N\to\infty} N^{-1}\sum_{m<N}\left|p_K^{-1}\sum_{n<p_K} u(n+mp_K)\overline{w_n(\alpha)}\right|^2\\
&=\lim_{N\to\infty} N^{-1}\sum_{m<N}\left|p_K^{-1}\sum_{n<p_K} \overline{w_n(\mathscr{C}_{K,\alpha})}u(n+mp_K)\right|^2\\
&=\lambda^u(\mathscr{C}_{K,\alpha})
\end{aligned}$$

and $\lim_{K\to\infty}\lambda^u(\mathscr{C}_{K,\alpha})=\lambda^u(\{\alpha\})$. $\square$

Corollary. *If $\lambda^u(\{0\})=0$, then u has a mean equal to zero, i.e.*

$$\lim_{N\to\infty} N^{-1}\sum_{n<N} u_n=0.$$

2.2. Theorem. λ^u *is continuous if and only if* $\lim_{N\to\infty} N^{-1}\sum_{k<N}|\gamma^u(k)|^2=0$.

Proof

$$N^{-1}\sum_{k<N}|\gamma^u(k)|^2=N^{-1}\sum_{k<N}\int_{\hat{G}_q^2} w_k(x\ominus y)\lambda^u\otimes\lambda^u(\mathrm{d}x,\mathrm{d}y).$$

Put $\phi_N(x,y)=N^{-1}\sum_{k<N} w_k(x\ominus y)$. Then $|\phi_N|\leqslant 1$ and ϕ_N converges pointwise to the characteristic function of the diagonal Δ of $\hat{G}_q^2$, since $\lim_{N\to\infty} N^{-1}\sum_{k<N} w_k(\alpha)=\delta_0(\{\alpha\})$ ($\alpha\in\hat{G}_q$). From the dominated convergence theorem, we get

$$\lim_{N\to\infty} N^{-1}\sum_{k<N}|\gamma^u(k)|^2=\int_\Delta \lambda^u\otimes\lambda^u(\mathrm{d}x,\mathrm{d}y)=\sum_{\alpha\in E}\lambda^u(\{\alpha\})^2,$$

where E denotes the countable set of the elements α of $\hat{G}_q$ such that $\lambda^u(\{\alpha\})>0$. Hence the conclusion. □

Definition. If the Walsh spectral measure of $u \in \mathcal{S}_q$ is continuous, we shall say that u is *pseudo-random in the sense of Walsh.*

Remark. If u is pseudo-random in the sense of Walsh, u has a mean equal to zero on each arithmetical progression with difference p_K ($K \in \mathbb{N}$), given that for each $\alpha \in \hat{G}_q$ we have $\lim_{N\to\infty} N^{-1}\sum_{n<N} u_n\overline{w_n(\alpha)}=0$, and we can combine these equalities using the following remark:

$$p_K^{-1} \sum_{\alpha=(\alpha_0,\ldots,\alpha_{K-1},0,\ldots)} w_a(\alpha)\overline{w_n(\alpha)} = \begin{cases} 1 & \text{if } a \equiv n[p_k], \\ 0 & \text{in the other case.} \end{cases}$$

Theorems 2.1 and 2.2 should be compared to the one of classical spectral analysis [13]. The following theorem is analogous to the Van der Corput theorem.

2.3. Theorem. *If a sequence u of real numbers is such that for each $k \in \mathbb{N}^*$, the sequence $(u(n\oplus k)-u(n))_{n\in\mathbb{N}}$ is uniformly distributed* mod 1, *then u is uniformly distributed* mod 1.

Proof. The Weyl criterion (i.e. (x_n) is uniformly distributed if and only if $\lim_N N^{-1}\sum_{n<N} \mathrm{e}(jx_n)=0$ for all $j\neq 0$) implies that for each $j\in\mathbb{Z}^*$, the sequence $v_j: n\to \mathrm{e}(ju_n)$ has a Walsh correlation equal to zero on $\mathbb{N}^*$. Thus its Walsh spectral measure is the Haar measure h, and v_j is pseudo-random in the sense of Walsh and has a mean equal to zero. We deduce the conclusion from the Weyl criterion. □

In the following proposition the notations are the same as in Theorem 1.1.

Proposition. *Let u be an element of $\bigcap_{N\in\mathbb{N}} \mathcal{S}_{b_N}$ and $\gamma^u_{(N)}$ its correlation with respect to G_{b_N}.*

(1) *u belongs to the Wiener set of sequences g of complex numbers, i.e. such a g that the limit* $\lim_{N\to\infty} N^{-1}\sum_{n<N} g(n+k)\overline{g(n)}$ *exists for each integer k* [2].

(2) *The sequence $(\gamma^u_{(N)})_{N\in\mathbb{N}}$ converges pointwise to the restriction to $\mathbb{N}$ of the usual correlation Γ^u of u.*

Proof. Let k be a natural number, let p be such that $b_p>k$ and let N be an element of $\mathbb{N}$ written in base b_p:

$$N=n_0+n_1+\cdots+n_lb_p^l,\quad \text{with } n_i<b_p \text{ for each } i\in\{1,\ldots,l\}.$$

Then,

$$\left|N^{-1}\sum_{n<N} u(n\oplus_{b_p} k)\overline{u(n)} - N^{-1}\sum_{n<N} u(n+k)\overline{u(n)}\right|$$
$$\leq N^{-1}\cdot 2kb_p^{l-1}n_l \leq b_p^{-1}\cdot 2k.$$

Hence $N^{-1}\sum_{n<N} u(n\oplus_{b_p} k)\overline{u(n)}$ converges uniformly to $N^{-1}\sum_{n<N} u(n+k)\overline{u(n)}$ when p tends to ∞, which yields the existence of $\lim_{N\to\infty} N^{-1}\sum_{n<N} u(n+k)\overline{u(n)}$

and the equality

$$\lim_{N\to\infty} N^{-1} \sum_{n<N} u(n+k)\overline{u(n)} = \lim_{p\to\infty} \gamma^{u}_{(p)}(k). \qquad \square$$

Then we deduce our following theorem from Theorem 1.1.

2.4. Theorem. *Let u be an element of $\bigcap_{N\in\mathbb{N}} \mathcal{S}_{b_N}$ and $\lambda^{u}_{(N)}$ its spectral measure in $\hat{G}_{b_N}$. The sequence $(\Psi_{b_N} \cdot \lambda^{u}_{(N)})_{n\in\mathbb{N}}$ converges weakly to the classical spectral measure Λ^u of u.*

3. Application to the q'-multiplicative sequences

3.1. General case

Our aim is now to show that each q'-multiplicative sequence belongs to $\mathcal{S}_q$: Let $q=(q_0, q_1, \ldots)$ and $q'=(q'_0, q'_1, \ldots)$ be two sequences of natural numbers with $q_0 = q'_0 = 1$ and $q_i \geqslant 2$, $q'_i \geqslant 2$ for $i \neq 0$. Let u be a q'-multiplicative sequence with $|u|=1$. We know that u is in the Wiener space, and we will denote by Γ^u its usual correlation, and by Λ^u its usual spectral measure.

Notations. (1) For each natural number $t \neq 0$, put $D(t) = \{n \oplus_q t - n;\ n \in \mathbb{N}\}$. If $t = t_{r_1} p_{r_1} + \cdots + t_{r_c} p_{r_c}$, with $r_1 < \cdots < r_c$ and $t_{r_i} \in \{1, \ldots, q_{r_i+1} - 1\}$, we have

$$\begin{aligned} D(t) &= \{n \oplus_q t - n;\ n < p_{r_c+1}\} \\ &= \{\varepsilon_1 p_{r_1} + \cdots + \varepsilon_c p_{r_c};\ \varepsilon_i = t_{r_i} \text{ or } t_{r_i} - q_{r_i+1},\ \forall i \leqslant c\}, \end{aligned}$$

whence card $D(t) = 2^{c(t)}$ where $c(t)$ is the number of nonzero digits of t in base q.

(2) Suppose that $t < p_k$, and let y be an element of $D(t)$, $y = \varepsilon_1 p_{r_1} + \cdots + \varepsilon_c p_{r_c}$. Then $\{n \in \mathbb{N};\ n \oplus_q t = n + y\}$ is a union of arithmetic progressions with difference p_k. Let us denote $D_t(y) = \{n < p_k;\ n \oplus_q t = n + y\}$.

(3) If $t \in \mathbb{N}^*$, $t < p_k$, then for all large l we will denote $\mathrm{HCF}(p_k, p'_l) = \delta$.

3.1. Theorem. *Let u be a q'-multiplicative sequence with $|u|=1$; then u is an element of $\mathcal{S}_q$.*

Proof. To prove that u has a Walsh correlation, it is sufficient to prove, for each element t of $\mathbb{N}^*$ and each element $y > 0$, $y \in D(t)$, the existence of the following limit:

$$\lim_{N\to\infty} N^{-1} \sum_{\substack{n \equiv b[p_k] \\ n<N}} u(n+y)\overline{u(n)}, \quad \text{with } t < p_k \text{ and } y = b \oplus_q t - b,$$

as we then have

$$\begin{aligned} &\lim_{N\to\infty} N^{-1} \sum_{n<N} u(n \oplus_q t)\overline{u(n)} \\ &\quad = \sum_{\substack{y\in D(t) \\ y>0}} \Bigg(\sum_{b\in D_t(y)} \lim_{N\to\infty} N^{-1} \sum_{\substack{n<N \\ n\equiv b[p_k]}} u(n+y)\overline{u(n)} \\ &\qquad\qquad + \sum_{t\in D_{\ominus t}(p_{r_c+1}-y)} \lim_{N\to\infty} N^{-1} \sum_{\substack{n<N \\ n\equiv b[p_k]}} \overline{u(n+p_{r_c+1}-y)}u(n) \Bigg). \end{aligned}$$

In the sequel, y will be an element of $D(t)$, $y>0$, with $t<p_k$ and $b\in D_t(y)$.

(2) Consider the q'-multiplicative sequence u_l defined by $u_l(n)=\prod_{r<l} u(e'_r(n)p'_r)$ where $(e'_r(n))_{n\in\mathbb{N}}$ is the sequence of the digits of n in base q'. We shall denote $\delta_l=\mathrm{HCF}(p_k,p'_l)$; for l large enough, δ_l is constantly equal to δ and in the following we will suppose $\delta_l=\delta$. The sequence of numbers $u_l(n+y)\overline{u_l(n)}$ for which $n\equiv b[p_k]$ is periodic with periodicity $p'_l\delta^{-1}$. Hence, when N tends to $+\infty$,

$$N^{-1}\sum_{\substack{n<N\\ n\equiv b[p_k]}} u_l(n+y)\overline{u_l(n)}$$

$$=N^{-1}E(p_k^{-1}(N-b))\left[E(p_k^{-1}(N-b))^{-1}\sum_{\substack{n<N\\ n\equiv b[p_k]}} u_l(n+y)\overline{u_l(n)}\right]$$

converges to $\delta\cdot p_k^{-1}\cdot p_l'^{-1}\sum_{n<p'_l,\delta|(b-n)} u_l(n+y)\overline{u_l(n)}$. Let us denote $\sigma_{b,l}=\sum_{n<p'_l,n\equiv b[\delta]} u_l(n+y)\overline{u_l(n)}$. We have

$$\lim_{N\to\infty}\sum_{\substack{n<N\\ n\equiv b[p_K]}} u_l(n+y)\overline{u_l(n)}=\delta p_K^{-1}p_l'^{-1}\sigma_{b,l}.$$

(3) Given that $y>0$, we also have

$$\left|N^{-1}\sum_{\substack{n<N\\ n\equiv b[p_K]}} u_l(n+y)\overline{u_l(n)}-N^{-1}\sum_{\substack{n<N\\ n\equiv b[p_K]}} u(n+y)\overline{u(n)}\right|$$

$$\leqslant 2N^{-1}\mathrm{card}\{n<N;\ E(p_l'^{-1}(n+y))>E(p_l'^{-1}n)\}\leqslant p_l'^{-1}\cdot 2y.$$

In view of this estimation, we obtain that the following limits exist and are equal:

$$\lim_{N\to\infty} N^{-1}\sum_{\substack{n<N\\ n\equiv b[p_K]}} u(n+y)\overline{u(n)}=\lim_{l\to\infty}\delta p_K^{-1}p_l'^{-1}\sigma_{b,l}$$

and u has a Walsh correlation.

(4) We can precise $\lim_{l\to\infty}\delta p_K^{-1}p_l'^{-1}\sigma_{b,l}$: Let H be a positive integer such that $H\leqslant l$, $\delta|p'_H$ and $p_K<p'_H$. The euclidian division of n by p'_H gives

$$p_l'^{-1}\sigma_{b,l}$$

$$=p_l'^{-1}\sum_{\substack{n<p'_l\\ n\equiv b[\delta]}} u_l(n+y)\overline{u_l(n)}$$

$$=p_l'^{-1}\sum_{n<p'_l(p'_H)^{-1}}\Bigg(\sum_{\substack{r\equiv b[\delta]\\ r<(p'_H)-y}} u_l(r+y)\overline{u_l(r)}$$

$$+\sum_{(p'_H)-y\leqslant r<p'_H} u_l((n+1)p'_H)\overline{u_l(np'_H)}u_l(r+y-p'_H)\overline{u_l(r)}\Bigg)$$

$$=p_H'^{-1}\sum_{\substack{r\equiv b[\delta]\\ r<(p'_H)-y}} u(r+y)\overline{u(r)}$$

$$+\sum_{\substack{(p'_H)-y\leqslant r<p'_H\\ r\equiv b[\delta]}} u(r+y-p'_H)\overline{u(r)}\left(p_l'^{-1}\sum_{n<p'_l(p'_H)^{-1}} u_l((n+1)p'_H)\overline{u_l(np'_H)}\right).$$

But

$$\lim_{l\to\infty} p_l'^{-1} \sum_{n<p_l'p_H'^{-1}} u_l((n+1)p'_H)\overline{u_l(np'_H)}$$

$$= \lim_{l\to\infty} p_l'^{-1} \sum_{q'_H+1\leq n<(p'_H)^{-1}p'_l} u_l(p'_H) = p_H'^{-1}u(p'_H).$$

Finally, we obtain the following expression with only finite sums:

$$\lim_{N\to\infty} N^{-1} \sum_{\substack{n<N\\ n\equiv b[\delta]}} u(n+y)\overline{u(n)}$$

$$= p_K^{-1}p_H'^{-1}\delta\Bigg[\sum_{\substack{r<p'_H-y\\ r\equiv b[\delta]}} u(r+y)\overline{u(r)} + \sum_{\substack{p'_H-y\leq r<p'_H\\ r\equiv b[\delta]}} u(r+y-p'_H)u(p'_H)\overline{u(r)}\Bigg]. \quad \square$$

3.2. Case of the q-multiplicative sequences $(q = q')$

Let u be a q-multiplicative sequence such that $|u| = 1$. We know that u belongs to $\mathscr{S}_q$; in this case γ^u has a simpler expression: Let t and K be two natural numbers such that $t<p_K$. The sequence $(u(n\oplus t)\overline{u(n)})_{n\in\mathbb{N}}$ is periodic with periodicity p_K, and we have

$$\gamma^u(t) = p_K^{-1} \sum_{n<p_K} u(n\oplus t)\overline{u(n)}.$$

More generally, if a is a natural number such that $a<q_{K+1}$, then

$$\gamma^u(t+ap_K) = p_{K+1}^{-1} \sum_{n<p_K} u(n\oplus(t+ap_K))\overline{u(n)}$$

$$= p_K^{-1}q_{K+1}^{-1} \sum_{n<p_K} (u(n\oplus(t+ap_K))\overline{u(n)} + \cdots$$

$$+ u(n+(q_{K+1}^{-1})p_K \oplus (t+ap_K))\overline{u(n+(q_{K+1}^{-1})p_K)})$$

$$= \gamma^u(t)\left(q_{K+1}^{-1} \sum_{i<q_{K+1}} u(ap_K\oplus ip_K)\overline{u(ip_K)}\right).$$

Thus we have proved the following theorem.

3.2. Theorem. (1) *Let u be a q-multiplicative sequence such that* $|u| = 1$. *Its Walsh correlation is the q-multiplicative sequence defined by*

$$\gamma^u(ap_K) = q_{K+1}^{-1} \sum_{i<q_{K+1}} u((a\oplus i)p_K)\overline{u(ip_K)}$$

for each integer K and a with $a<q_{K+1}$.

(2) *If we denote* $\rho_{a,k} = \gamma^u(ap_k)$, *the Walsh spectral measure* λ^u *of* u *is the infinite convolution product*:

$$\mathop{*}_{k=0}^{\infty} \sum_{j<q_{k+1}} \left[q_{k+1}^{-1} \sum_{a<q_{k+1}} \rho_{a,k} \overline{w_{ap_k}(\varepsilon_k^j)} \right] \delta_{\varepsilon_k^j}$$

where ε_k^j *is the element of* $\hat{G}_q$ *defined by*

$$\varepsilon_k^j(n) = \begin{cases} j & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

Denote

$$H_j^{(k)} = q_{k+1}^{-1} \sum_{a<q_{k+1}} \rho_{a,k} \overline{w_{ap_k}(\varepsilon_k^j)} = \left| q_{k+1}^{-1} \sum_{b<q_{k+1}} u(bp_k) \overline{\zeta_{k+1}^{jb}} \right|^2,$$

$$r_k = \max_{j<q_{k+1}} |H_j^{(k)} - q_{k+1}^{-1}|,$$

and assume that $\sup q_n < +\infty$. Then from Theorem 3.2 and [1] we get

(a) if $\sum_k (1 - \max_{j<q_{k+1}} H_j^{(k)})$ converges, then λ^u is discrete;

(b) in the other case, either there exists an integer k such that $\sum_n r_n^{2k}$ converges and μ is absolutely continuous with respect to the Haar measure h of $\hat{G}_q$ (and $\mu \in L^2(h)$), or μ is singular continuous.

3.3. Example. $u(n) = \mathrm{e}(n\alpha)$, where $\alpha \in \mathbb{R}$, and $q = (1, q, q, \ldots)$. u is q-multiplicative and its Walsh correlation is given by

$$\begin{aligned} \rho_{a,k} &= q^{-1} \sum_{i<q} u((a \oplus i)q^k) \overline{u(iq^k)} \\ &= q^{-1}[(q-a)u(aq^k) + a\overline{u((q-a)q^k)}]. \end{aligned}$$

Denote $\zeta = \mathrm{e}(q^{-1})$. With the previous notations, we have

$$H_j^{(k)} = \begin{cases} [q^{-1} \sin^{-1}(q^{-1}\pi j - \pi q^k \alpha) \sin(\pi q^{k+1}\alpha)]^2 & \text{if } \zeta^j \neq \mathrm{e}(\alpha q^k), \\ 1 & \text{if } q^k \alpha = q^{-1} j \bmod 1. \end{cases}$$

Denote $E_q = \{x \in \mathbb{Q};\ \exists k \in \mathbb{N},\ xq^k \in \mathbb{Z}\}$; in the case $\alpha \notin E_q$ we are going to study the series

$$\sum_k \left[1 - \sin^2(\pi q^{k+1}\alpha) q^{-2} \left(\min_{j<q} \sin^2(q^{-1}\pi j - \pi q^k \alpha) \right)^{-1} \right].$$

Put $\alpha = \alpha_0 q^{-1} + \alpha_1 q^{-2} + \cdots + \alpha_{k+1} q^{-k-2} + \cdots$ ($\in]0, 1[$), $\beta_k = \alpha_{k+1} q^{-2} + \alpha_{k+2} q^{-3} + \cdots$ ($\in]0, q^{-1}[$ because $\alpha \notin E_q$), and

$$A_k = \sin^2(\pi q^{k+1}\alpha) q^{-2} (\min_{j<q} \sin^2(q^{-1}\pi j - \pi q^k \alpha))^{-1}.$$

$$A_k \in \{q^{-2} \sin^{-2} \pi\beta_k \sin^2 \pi q \beta_k,\ q^{-2} \sin^{-2}(\pi q^{-1} - \pi\beta_k) \sin^2 \pi q \beta_k\}.$$

For infinitely many k (for which $\alpha_k \neq q-1$ or $\beta_k \leq 2^{-k}$), A_k can be written as $q^{-2} \sin^{-2} x \sin^2 qx$ with $x \in]0, \frac{1}{2}q^{-1}\pi]$. Let N_0 be the set of these values of k. The function $f: x \in]0, \frac{1}{2}q^{-1}\pi] \to q^{-2}\sin^{-2} x \sin^2 qx$ decreases from 1 to $q^{-2}\sin^{-2}(\frac{1}{2}q^{-1}\pi)$. The general term of the studied series tends to 0, i.e. A_k tends to 1 only if

$$\forall \varepsilon > 0,\ \exists K \in \mathbb{N},\ \forall k > K,\ k \in N_0, \quad \beta_k \in]0, \varepsilon[\cup]q^{-1} - \varepsilon, q^{-1}[. \qquad (*)$$

Let $\varepsilon_1 < q^{-2}$ and K_1 such that $(*)$ holds $(q^{-1} - \varepsilon_1 > q^{-2}(q-1))$. Let $k \in N_0$, $k > K_1$; $\beta_k \in]0, q^{-2}[\cup]q^{-2}(q-1), q^{-1}[$, whence $\alpha_{k+1} \in \{0, q-1\}$. Therefore, if A_k tends to 1, for $k > k_0$, $\alpha_k \in \{0, q-1\}$. Taking into account that $\alpha \notin E_q$, there is an infinity of positive integers k for which $\alpha_{k+1} = 0$ and $\alpha_{k+2} = q-1$. We get $q^{-3}(q-1) < \beta_k < q^{-2}$, whence a contradiction with $\lim A_k = \lim f(\pi \cdot \beta_k) = 1$. Thus, the series considered diverges.

In the case where $\alpha \notin E_q$, we can easily check that r_k does not tend to 0. Consequently,

(a) on the assumption that $\alpha \in E_q = \{x \in \mathbb{Q}; \exists k \in \mathbb{N}, xq^k \in \mathbb{Z}\}$ for $k > k_0$, all the $\Pi_j^{(k)}$ are equal to zero, and we find that

$$\lambda^u = \mathop{*}_{k=0}^{\infty} \sum_{j<q} \Pi_j^{(k)} \delta(\varepsilon_k^j)$$

is discrete, its support being a finite set of elements of $\hat{G}_q$ with only a finite number of nonzero coordinates;

(b) if $\alpha \notin E_q$, λ^u is continuous and singular.

3.4. Example (*Van der Corput sequence*). Let $n = \sum d_k(n) q^k$ be a natural number written in a constant base $q = (1, q, q, \ldots)$. We denote $v_n^{(q)} = \sum_{k \geq 0} d_k(n) q^{-(k+1)}$. The sequence $u(n) = \mathrm{e}(v^{(q)}(n))$ is q-multiplicative. By making a computation as in the Example 3.3, we obtain

$$\rho_{a,k} = q^{-1}[(q-a)\,\mathrm{e}(aq^{-(k+1)}) + a\,\mathrm{e}((a - q^{k+1})q^{-(k+1)})],$$

$$\Pi_j^{(k)} = \begin{cases} (q^{-1} \sin q^{-k}\pi \sin^{-1}(q^{-1}\pi j - q^{-k-1}\pi))^2 & \text{if } (k,j) \neq (0,1), \\ 1 & \text{if } (k,j) = (0,1). \end{cases}$$

The series $\sum_{k>0} (1 - \max_{j<q} q^{-2} \sin^2 q^{-k}\pi \sin^{-2}(q^{-1}\pi j - q^{-k-1}\pi))$ converges and λ^u is discrete.

3.5. Example. $u(n) = \mathrm{e}(\beta s_q(n))$, where $\beta \in \mathbb{R}$ and $s_q(n)$ is the sum of the digits of n in the constant base q. We get

$$\rho_{a,k} = q^{-1}[(q-a)\,\mathrm{e}(\beta a) + q\,\overline{\mathrm{e}(\beta(q-a))}] \quad \text{and}$$

$$\Pi_j^{(k)} = \begin{cases} q^{-2} \sin^2 q\pi\beta \sin^{-2}(q^{-1}\pi j - \pi\beta) & \text{if } \mathrm{e}(\beta) \neq \zeta^j, \\ 1 & \text{if } \beta \equiv q^{-1} j \bmod 1. \end{cases}$$

$\rho_{a,k}$ and $\Pi_j^{(k)}$ do not depend on k. We obtain

(a) if $\exists j_0 \in \{0, \ldots, q-1\}$ such that $\beta \equiv q^{-1} j_0$, then $\lambda^u = \delta(j_0, j_0, \ldots)$;

(b) in the other case, either $q = 2$ and $\beta \equiv \frac{1}{4} \bmod \frac{1}{2}$, and λ^u is absolutely continuous, or λ^u is singular continuous.

3.3. Case of the q'-multiplicative sequences with $(q'_i, q_j) = 1$, for each i and j.

Proposition. *Let u be a q'-multiplicative sequence with constant modulus equal to* 1, *and such that $(q_i, q'_j) = 1$ for any integers i and j. The Walsh correlation of u with respect to G_q is given by*

$$\gamma^u(t) = p_k^{-1} \sum_{y \in D(t)} \operatorname{card} D_t(y) \Gamma^u(y) \quad (\textit{with } t < p_k),$$

where Γ^u is the ordinary correlation of u.

Proof. It follows from the general expression of γ^u and from

$$\lim_{l \to \infty} p_l'^{-1} \sigma_{b,l} = \Gamma^u(y) \quad \text{(cf. proof of Theorem 3.1).} \qquad \square$$

Corollary. *Let u be a q'-multiplicative sequence with constant modulus equal to* 1, *and such that $(q_i, q'_j) = 1$ for any integers i and j. If u is pseudo-random, u is pseudo-random in the sense of Walsh.*

Proof

$$\begin{aligned}
\sum_{t < p_k} |\gamma(t)| &\leq \sum_{t < p_k} \sum_{y \in D(t)} p_k^{-1} \operatorname{card} D_t(y) |\Gamma^u(y)| \\
&= |\Gamma^u(0)| + \sum_{1 \leq y < p_k} |\Gamma^u(y)| \\
&\quad \times \Bigg(\sum_{\substack{t < p_k \\ y \in D(t)}} p_k^{-1} \operatorname{card} D_t(y) + \sum_{\substack{t < p_k \\ -y \in D(t)}} p_k^{-1} \operatorname{card} D_t(-y) \Bigg) \\
&= |\Gamma^u(0)| + \sum_{1 \leq y < p_k} |\Gamma^u(y)| \Bigg(\sum_{n < p_k - y} p_k^{-1} + \sum_{y \leq n < p_k} p_k^{-1} \Bigg) \\
&\leq |\Gamma^u(0)| + 2 \sum_{1 \leq y < p_k} |\Gamma^u(y)|
\end{aligned}$$

since $t < p_k$, and $y \in D(t)$ if there is a positive integer n in $D_t(y)$ such that $t \oplus n = y + n < p_k$. The computations are the same for $-y$ ($|\Gamma^u(y)| = |\Gamma^u(-y)|$).

From the Cauchy-Schwarz inequality we get

$$p_k^{-1} \sum_{t < p_k} |\gamma^u(t)| \leq 2 \Bigg(p_k^{-1} \sum_{t < p_k} |\Gamma^u(y)|^2 \Bigg)^{1/2},$$

which tends to 0 when k tends to $+\infty$. Hence,

$$\lim_{N \to \infty} N^{-1} \sum_{t < N} |\gamma(t)|^2 = 0.$$

Theorem 2.2 yields the conclusion. $\square$

If the ordinary spectral measure of u is discrete, its support is a translation of the subgroup $\mathbb{T}'_q$ generated by the $p_k'^{-1}$ mod 1, denoted $\alpha + \mathbb{T}_{q'}$: $\Lambda^u = \sum_{k=0}^{\infty} c_k \delta_{\alpha + x_k}$ where $\{x_k\,;\, k \in \mathbb{N}\} \subset \mathbb{T}_{q'}$, $c_k \in [0, 1]$ and $\sum_{k=0}^{\infty} c_k = 1$.

3.6. Theorem. *Let u be a q'-multiplicative sequence with constant modulus equal to 1, and such that $(q_i, q'_j) = 1$ for any integers i and j. If the spectral measure of u is a discrete measure $\Lambda^u = \sum_{k=0}^{\infty} c_k \delta_{\alpha+x_k}$ where $\{x_k ; k \in \mathbb{N}\} \subset \mathbb{T}_{q'}$, and $c_k \in [0, 1]$, $\sum_{k=0}^{\infty} c_k = 1$, then its Walsh spectral measure is $\lambda^u = \sum c_k \mu_k$ where μ_k is the Walsh spectral measure of the sequence $n \to \mathrm{e}(n(\alpha + x_k))$.*

Proof. If $t < p_k$,

$$\begin{aligned}\gamma(t) &= p_k^{-1} \sum_{y \in D(t)} \operatorname{card} D_t(y) \Gamma^u(y) \\ &= \sum_{k \geqslant 0} c_k \sum_{y \in D(t)} p_k^{-1} \operatorname{card} D_t(y)\, \mathrm{e}(y(\alpha + x_k)) = \sum_{k \geqslant 0} c_k \gamma_k(t)\end{aligned}$$

where γ_k denotes the Walsh correlation of the sequence $n \to \mathrm{e}(n(\alpha + x_k))$. $\square$

Example. If q and q' are two integers $\geqslant 2$ such that $(q, q') = 1$, and if u is a q'-multiplicative sequence such that $|u| = 1$, we have

(a) if $\Lambda^u(\{x\}) = 0$, for each element x of E_q, the Walsh spectral measure of u is continuous.

(b) if $\Lambda^u(\{\alpha\}) = c > 0$ the Walsh spectral measure of u is $c\mu + (1 - c)\mu'$ where μ is a discrete measure on a finite subset of elements of $\hat{G}_q$ with a finite number of nonzero coordinates and μ' is continuous.

Remarks. (1) The corollary is untrue in the general case (cf. examples with $q = q'$).

(2) The reciprocal of the corollary is false (in the case $q = 2$ and q' odd, we can have Λ^u discrete with $\Lambda^u(\{x\}) = 0$ for each element x of E_2; then λ^u is continuous singular: take for instance $u(n) = \mathrm{e}(\beta n)$ with $\beta \notin E_2$).

References

[1] G. Brown and W. Moran, Coin tossing and powers of singular measures, *Math. Proc. Camb. Phil. Soc.* 77 (1975) 349-364.

[2] J. Coquet, Contribution à l'étude harmonique de suites arithmétiques, Thèse d'Etat, Orsay (1978).

[3] J. Coquet, Répartition modulo un des suites q-additives, *Comment. Math.* **21** (1979) 23-42.

[4] J. Coquet, Mesures spectrales de Walsh associées à certaines suites arithmétiques, *Ann. Inst. Fourier Grenoble* **35** (1985) 1-12.

[5] R. Ellis, *Lectures on Topological Dynamics* (Benjamin, New York, 1969).

[6] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis. Vol. 1* (Springer, Berlin, 1963).

[7] T. Kamae, Mutual singularity of spectra of dynamical systems given by "sums of digits" to different bases, *Bull. Soc. Math. Fr.* **49** (1977) 109-114.

[8] P. Liardet, Propriétés dynamiques de suites arithmétiques, dans: *Journées Mathématiques SMF-CNRS, Théorie Analytique et Elémentaire des Nombres*, Caen (1980) 58-73.

[9] P. Liardet, Regularities of distribution, *Compositio Math.* **61** (1987) 267-293.

[10] P. Liardet, Propriétés harmoniques de la numération selon J. Coquet, *Lecture Notes in Mathematical Comp. Math.* **61** (1987) 267-293.

[11] W. Parry, *Topics in Ergodic Theory* (Cambridge University Press, 1981).

[12] M. Queffélec, Mesures spectrales associées à certaines suites arithmétiques, *Bull. Soc. Math. France* **107** (1979) 385-421.

[13] G. Rauzy, *Propriétés Statistiques de Suites Arithmétiques* (P.U.F., Collection Sup, 1976).

[14] J. L. Walsh, A closed set of normal orthogonal functions, *Amer. J. Math.* **55** (1923) 5-24.

[15] N. Wiener, *The Fourier Integral and Certain of its Applications* (Cambridge University Press, 1988).